ISRAEL JOURNAL OF MATHEMATICS 153 (2006), 307–318

TWO MAXIMAL SUBGROUPS OF $E_8(2)$

BY

CHRIS PARKER

School of Mathematics, University of Birmingham
Edgbaston, Birmingham B15 2TT, UK
e-mail: C.W.Parker@bham.ac.uk

AND

JAN SAXL

Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge CB3 0WB, UK
e-mail: J.Saxl@dpmms.cam.ac.uk

ABSTRACT

We show that $E_8(2)$ has a unique conjugacy class of subgroups isomorphic to $PSp_4(5)$ and a unique conjugacy class of subgroups isomorphic to $PSL_3(5)$. There normalizers are maximal subgroups of $E_8(2)$ and are, respectively, isomorphic to $PGSp_4(5)$ and $Aut(PSL_3(5))$.

This paper grew out of our paper [13], which was concerned with localizations of finite simple groups. It turned out that $PSp_4(5)$ has a localization in $E_8(2)$. This is one context of our main result.

THEOREM 1:

(i) *There is a unique conjugacy class of subgroups* $PSp_4(5)$ *in* $E_8(2)$. *The normalizer of such a subgroup is isomorphic to* $PGSp_4(5)$ *and is a maximal subgroup in* $E_8(2)$.

(ii) *There is a unique conjugacy class of subgroups* $PSL_3(5)$ *in* $E_8(2)$. *The normalizer of such a subgroup is isomorphic to* $Aut(PSL_3(5))$ *and is a maximal subgroup in* $E_8(2)$.

This theorem can be viewed as a minor step in the investigation of conjugacy classes of cross-characteristic embeddings of groups of Lie type into finite exceptional simple groups of Lie type. An important starting point for that is the

Received September 22, 2004

paper of Liebeck and Seitz [12], which gives a list of embeddings. A detailed description of their conjugacy classes, as well as minimal fields over which they are defined, remains to be determined in many cases there. For example, it is mentioned in [12] that both $\mathrm{PSp}_4(5)$ and $\mathrm{PSL}_3(5)$ are subgroups of $\mathrm{E}_8(2^a)$ for suitable a and in particular for a even—however, the fact that the minimal a is one as well as uniqueness of the embeddings into $\mathrm{E}_8(2)$ up to conjugacy is new.

A starting point for our proof is a result in [3, Theorem 5.1] establishing the existence of subgroups $\mathrm{PSL}_4(5)$ in $\mathrm{E}_8(4)$. We require the following extension of this result.

THEOREM 2: *There is a unique conjugacy class of subgroups* $\mathrm{PSL}_4(5)$ *in* $\mathrm{E}_8(4)$. *The normalizer in* $\mathrm{E}_8(4)$ *of this group is a maximal subgroup of* $\mathrm{E}_8(4)$ *and is isomorphic to the extension* $\mathrm{PSL}_4(5).2$ *of the simple group by a graph automorphism which centralizes a subgroup isomorphic to* $\mathrm{PSp}_4(5)$.

It is easy to see that $\mathrm{PSL}_4(5)$ cannot be contained in $\mathrm{E}_8(2)$ (by comparing the 5-part of their orders, for example). However, the restriction of the adjoint module of $\mathrm{E}_8(4)$ to $\mathrm{PSL}_4(5)$ is an irreducible module which, by [7], can be realized over $\mathrm{GF}(2)$.

We shall obtain our subgroups in Theorem 1 as embedded in the centralizers of carefully chosen field automorphisms of $\mathrm{E}_8(4)$ which normalize $\mathrm{PSL}_4(5)$. We use properties of amalgams of subgroups, as well as detailed information on the 5-structure of $\mathrm{E}_8(2)$, to show that these groups are unique up to conjugacy.

We first collect the information on $\mathrm{E}_8(2)$ we shall need.

PROPOSITION 3: *Assume that* $G = \mathrm{E}_8(2)$, $S \in \mathrm{Syl}_5(G)$ *and* $Z = Z(S)$.

(i) $|Z| = 5$, $|S| = 5^5$ *and* $X = N_G(Z) \cong \mathrm{SU}_5(4).4 = 5{\cdot}\mathrm{U}_5(4).4$.

(ii) *There exists a subgroup* Y *of* G *with* $Y \cong \mathrm{PGU}_5(4).4 = \mathrm{U}_5(4).5.4$.

(iii) X *contains the monomial group* $5^{1+3} : \mathrm{Sym}(5)$ *and* Y *contains the monomial group* $5^{3+1} : \mathrm{Sym}(5)$.

(iv) S *has exponent* 5.

(v) S *contains a unique abelian normal subgroup* Q_i *of order* 5^i *for* $1 \le i \le 4$.

(vi) $N_G(Q_4)$ *is isomorphic to a subgroup* $5^4.(2_+^{1+4} \circ 4).\mathrm{Sym}(6)$.

(vii) *For* $i = 2, 3$, $C_G(Q_i) = Q_4$ *and* $N_G(Q_i) \le N_G(Q_4)$.

(viii) $N_G(Q_3)/Q_4 \cong 4 \times \mathrm{Sym}(5)$.

(ix) S *contains exactly five conjugacy classes of extraspecial subgroups of order* 125 *and every such subgroup contains* Q_2.

(x) *If* E *is an extraspecial subgroup of* G *of order* 125, *then* $N_G(E)' \cong 5_+^{1+2} : \mathrm{SL}_2(5)$.

(xi) *$N_G(Q_2)/Q_4 \cong \mathrm{GL}_2(5)$ and $N_G(Q_2)$ contains a unique conjugacy class of subgroups isomorphic to $5^2 : \mathrm{GL}_2(5)$. If A_1 and A_2 are such subgroups with $|A_1 \cap A_2|$ divisible by $5^3.4^2$, then $A_1 = A_2$.*

Proof: Parts (i) and (ii) are well-known and can be read from [4]. Part (iii) follows directly from the subgroup structure of $\mathrm{SU}_5(4)$ and $\mathrm{PGU}_5(4)$. From (iii), we see that S is a semidirect product of a normal elementary abelian subgroup of order 5^4 and a cyclic group of order 5. Thus S is isomorphic to a subgroup of a Sylow 5-subgroup of $\mathrm{SL}_5(5)$ and consequently has exponent 5. Therefore, (iv) holds.

Assume without loss of generality that S is contained in the monomial group $X_1 = 5^{1+3} : \mathrm{Sym}(5)$. Put $Q_4 = O_5(X_1)$. Then Q_4 is abelian and normal in S. Furthermore, as $|Z| = 5$, Q_4 is the unique abelian subgroup of S of order 5^4. Since, as a GF(5)-module for Sym(5), Q_4 is isomorphic to a submodule of the 5-dimensional permutation module over GF(5), we see that Q_4 is uniserial as an S-module. Thus $Q_3 = [Q_4, S]$ has order 125, $Q_2 = [Q_3, S]$ has order 25 and $Z = [Q_2, S]$ has order 5. Hence, for $1 \leq i \leq 4$, Q_i is the unique normal abelian subgroup of S of order 5^i. Hence (v) holds. For $i = 2, 3, 4$, define $N_i = N_G(Q_i)$.

For (vi) we refer to [11, Table 5.2].

To prove part (vii) it suffices to show that $C_G(Q_2) = Q_4$ and this is what we shall do. Since $Z \leq Q_2$, $C_G(Q_2) = C_{C_G(Z)}(Q_2)$. Now, by (i), $C_G(Z) = X' \cong \mathrm{SU}_5(4)$ and Q_2 can be identified therein with the subgroup of diagonal matrices $\langle \mathrm{diag}(\omega, \omega, \omega, \omega, \omega), \mathrm{diag}(1, \omega, \omega^2, \omega^3, \omega^4)\rangle$ where ω is an element of order 5 from GF(16). It follows that

$$C_{C_G(Z)}(Q_2) = \langle\, \mathrm{diag}(\omega, \omega^{-1}, 1, 1, 1), \mathrm{diag}(1, \omega, \omega^{-1}, 1, 1), \\ \mathrm{diag}(1, 1, \omega, \omega^{-1}, 1), \mathrm{diag}(1, 1, 1, \omega, \omega^{-1})\rangle.$$

Since this group has order 5^4, we have $C_G(Q_2) = Q_4$ as claimed.

Consider now (viii). By considering the monomial subgroup $Y_1 \cong 5^{3+1} : \mathrm{Sym}(5)$, we see that N_3/Q_4 contains $Y_1/Q_4 \cong \mathrm{Sym}(5)$. Furthermore, $Z(N_4/Q_4)$ induces a group of scalars of order 4 on Q_4 and so its preimage K normalizes Q_3 (and Q_2). Thus we have $N_3/Q_4 \geq KY_1/Q_4 \cong 4 \times \mathrm{Sym}(5)$.

If $N_3 > KY_1$, then because N_3 operates irreducibly on $O_2(N_4)/K_1$ and because Sym(5) is a maximal subgroup of Sym(6), either N_3 contains a factor isomorphic to Sym(6) or $N_3/Q_4 \geq O_2(N_4/Q_4)$. If $N_3/Q_4 \geq O_2(N_4/Q_4)$, then N_3 acts irreducibly on Q_4, whereas in fact it normalizes Q_3, a contradiction. Therefore, $N_3/K \cong \mathrm{Sym}(6)$, and again N_3 acts irreducibly on Q_4, a contradiction. Thus $N_3 = KY_1$ as claimed. This proves (viii).

Suppose that E is an extraspecial subgroup of S of order 5^3. Then $S = EQ_4$ and $E \cap Q_4$ is normalized by both Q_4 and E. Hence $E \cap Q_4 = Q_2$ and $Z = Z(E)$. Let $x \in S \setminus Q_4$. Then $\langle x, Q_2 \rangle$ is a non-abelian subgroup of S of order 125. Hence E is extraspecial of exponent 5. Since there are $5^5 - 5^4$ choices for x and since each extraspecial subgroup contains $5^3 - 5^2$ such elements, there are exactly 25 extraspecial subgroups E of S of order 125. Let E be such an extraspecial subgroup. If $E \trianglelefteq S$, then $[E, S] \le [S, S] \cap E \le Q_4 \cap E = Q_2$. But this means $Q_3 = [Q_4, S] \le Q_2$, a contradiction. Therefore, $[S : N_S(E)] = 5$ and so we conclude that (ix) holds.

From [9, Table 3.5 B] the group $SU_5(4)$ contains five conjugacy classes of maximal subgroups isomorphic to $5_+^{1+2} : SL_2(5)$. So (x) follows from (ix).

Set $N_2 = N_G(Q_2)$. Let $L \cong 5^2 : SL_2(5)$ be a subgroup of Y and let $E \in Syl_5(L)$. Then E is extraspecial of order 5^3. Hence $N_G(E)'/E \cong SL_2(5)$ by (x). Thus using (ix), by conjugating by an element from Y and then from $N_G(E)$ we may suppose that $O_5(L) = Q_2$. In particular, $N_2/Q_4 \ge LQ_4/Q_4 \cong SL_2(5)$. As in the proof of (vii) we see that $N_2 \ge KL$ and then that $N_3/Q_4 \cong 4 \circ SL_2(5)$ or $GL_2(5)$. Since $C_{Q_4}(S) = Z$, we infer that, as an LQ_4/Q_4-module, Q_4 is a non-split extension of two natural $SL_2(5)$-modules. Thus Q_2 is the unique subgroup of Q_4 of order 5^2 normalized by L and so $N_{N_4}(LQ_4) \le N_G(Q_2)$. We conclude that $N_G(Q_2)/Q_4 \cong GL_2(5)$ and the first part of (xi) is proven. Since the complements to Q_4/Q_2 in N_2/Q_2 are centralizers of involutions, the subgroups $5^2.GL_2(5)$ are determined uniquely up to conjugacy in G. Suppose that A_1 and A_2 are such complements and assume that $y \in N_G(Q_2)$ is such that $A_1^y = A_2$. Let B be the normalizer of a Sylow 5-subgroup in A_1. So $|B| = 5^3.4^2$ is uniquely determined by its order in A_1. Assume that $B \le A_2$. We are required to show that $A_1 = A_2$. Since $B \le A_2 = A_1^y$, $B^{y^{-1}} \le A_1$. Hence there exists $a \in A_1$ such that $B^{y^{-1}a} = B$. But $N_{N_G(Q_2)}(B) \le N_{N_G(Q_2)}(A_1) = A_1$ and so $y^{-1} \in A_1$ and (xi) is proved. ∎

LEMMA 4: *Suppose that $G = E_8(4)$ and $S \in Syl_5(G)$. Let $Z_2 = Z_2(S)$ be the second centre of S and $K = C_G(Z_2)$. Then $K \le S$ and K is a torus of G of order 5^8.*

Proof: Let $Z = Z(S)$ and $C = C_G(Z)$. Then $K \le C_G(Z)$. From [5, Table 4.7.3 B] we see that $C = (SU_5(4) \circ SU_5(4)).5 = 5{\cdot}(U_5(4) \times U_5(4)).5$. Let $M \le C$ be the product of the monomial subgroups from the two components C_1 and C_2 of C. So $M = 5^{1+(3+3)}.(Sym(5) \times Sym(5)).5$ and we assume, as we may, that $S \le M$. Let K^* be the unique elementary abelian subgroup of S of order 5^8.

Then K^* is a maximal torus of G and $K^* = O_5(M)$. Assume that $Z_2 \not\leq K^*$. Then, as $[K^*, Z_2] \leq Z$, $Z_2Z/Z \leq C_{C/Z}(K^*/Z) = K^*/Z$. Thus $Z_2 \leq K^*$. It follows that $C_C(Z_2) = K^*C_{C_1}(Z_2)C_{C_2}(Z_2)$. We calculate $C_{C_1}(Z_2)$ and $C_{C_1}(Z_2)$ exactly as in the proof of Proposition 3 (v). This proves $K^* = K = C_G(Z_2)$. ∎

Definition 1: Suppose that $R \cong \mathrm{PSL}_4(5)$ and let R_1, R_2 be the maximal parabolic subgroups of R containing a fixed Borel subgroup B of R corresponding to point and plane stabilizers in R. So $R_1 \cong 5^3 : \mathrm{PSL}_3(5) \cong R_2$. Then a pair of subgroups (P_1, P_2) of a group G is an amalgam of type $\mathrm{PSL}_4(5)$ provided, for $i = 1, 2$, P_i has the same chief factors as R_i and $P_1 \cap P_2$ has the same chief factors as $R_1 \cap R_2$.

Notice that an amalgam of type $\mathrm{PSL}_4(5)$ contained in a group X does not necessarily determine a subgroup of X isomorphic to $\mathrm{PSL}_4(5)$ and in fact this is not usually the case in the following lemma.

PROPOSITION 5: *Suppose that $G = \mathrm{E}_8(p^a)$ with a even and p a prime which is not 5; then G contains at most one conjugacy class of amalgams of type* $\mathrm{PSL}_4(5)$.

Proof: Suppose that (P_1, P_2) is an amalgam of type $\mathrm{PSL}_4(5)$ in $\mathrm{E}_8(p^a)$. For $i = 1, 2$, set $Q_i = O_5(P_i)$. Then $P_1 \cong 5^3 : \mathrm{PSL}_3(5)$ and, as $\mathrm{PSL}_3(5)$ is not isomorphic to a subgroup of the Weyl group of type E_8, P_1 is the normalizer of a non-toroidal elementary abelian subgroup of order 5^3. Using [3], we see that P_1 is uniquely determined up to conjugacy in G. Let $S \in \mathrm{Syl}_5(P_1)$ and $P_{12} = N_{P_1}(Z(S))$. Then P_{12} is uniquely determined up to conjugacy in P_1 and $P_{12} \cong 5_+^{1+(2+2)}.\mathrm{GL}_2(5)$. Let $Q_{12} = O_5(P_{12})$. Then Q_{12} is extraspecial and $Q_{12}/Z(Q_{12})$ admits P_{12}/Q_{12} with two non-central chief factors. Let Z be the preimage of $Z(P_{12}/Q_{12})$. Then Z acts on $Q_1/Z(S)$ as the scalar λ and on Q_{12}/Q_1 it induces λ^3. Thus the two non-central chief factors for P_{12} on Q_{12} are non-isomorphic and consequently P_{12} normalizes precisely two subgroups of Q_{12} of order 5^3. Since Q_1 and Q_2 are such subgroups, we see Q_2 is uniquely determined up to conjugacy by P_1. Now P_2 normalizes Q_2 and consequently Q_2 is non-toroidal and so $P_2 = N_G(Q_2)$ is uniquely determined by P_1 and furthermore $P_1 \cap P_2 = P_{12}$. This proves the proposition. ∎

In the proof of Theorem 5.1 in [3], it is shown that an amalgam (P_1, P_2) of type $\mathrm{PSL}_4(5)$ exists in $\mathrm{E}_8(p^a)$ whenever $\mathrm{E}_8(p^a)$ contains a subgroup $5^3.\mathrm{PSL}_3(5)$. It is further shown in [3] that the amalgam (P_1, P_2) generates a subgroup of

$E_8(p^a)$ isomorphic to $PSL_4(5)$ if and only if $p = 2$ and a is even. Notice that (P_2, P_1) is also an amalgam of type $PSL_4(5)$ and so by Proposition 5 there exists $g \in E_8(p^a)$ such that $P_2^g = P_1$ and $P_1^g = P_2$. Therefore, upon setting $G = E_8(4)$ and $H = \langle P_1, P_2\rangle$, we see that $g \in N_G(H)$ and, since $P_1 = N_G(O_5(P_1))$, we have $N_G(H) = \langle H, g\rangle$. Thus, if $p = 2$ and a is even, $N_G(H)$ contains a subgroup $H\langle g\rangle \cong PSL_4(5).2$ where g conjugates P_1 to P_2. In particular, g does not induce a diagonal automorphism on H. Let τ be the inverse transpose automorphism of $PSL_4(5)$. Then the outer automorphism group of $PSL_4(5)$ is covered by $\langle \mathrm{diag}(i,1,1,1), \tau\rangle \cong \mathrm{Dih}(8)$. Hence up to conjugacy by elements of $\mathrm{Out}(PSL_4(5))$ we have $g \in \tau\, PSL_4(5)$ or $g \in \tau\, \mathrm{diag}(i,1,1,1)\, PSL_4(5)$. In the first case we have there is an involution t in $\tau\, PSL_4(5)$ such that t centralizes a subgroup of $PSL_4(5)$ isomorphic to $PSp_4(5)$. There are no such involutions in $\tau\, \mathrm{diag}(i,1,1,1)\, PSL_4(5)$ (see [5, 4.5.1]).

LEMMA 6: *Suppose that $G = E_8(4)$, $H \leq G$ with $H \cong PSL_4(5).2$. Then there is an involution t of H such that $C_H(t)' \cong PSp_4(5)$.*

Proof: Let $S \in \mathrm{Syl}_5(H)$ and Q be the unique elementary abelian subgroup of S of order 5^4. As we have already witnessed the non-trivial outer automorphism in H/H' is not diagonal. Using [5, Table 4.5.1], if s is an involution in H and $|C_H(s)|_5 \geq 5^3$, then $C_H(s)' \cong PSp_4(5)$. So to prove the lemma it suffices to show that there is such an involution. We do this in two steps. First suppose that there is an involution $t \in N_H(S)$ such that $|[S/Q, t]| = 5$. Let $N = N_H(Q) \cong 5^4.(SL_2(5) \circ SL_2(5)).2^2$ with $N_H(S) \leq N$ and let $z \in N$ be such that $C_N(z)$ is a complement to Q. Let T be a Sylow 2-subgroup of $N_H(S) \cap N$ containing t. Notice that $|T| = 2^5$. Since $[S,t] \not\leq Q$, we see that $|C_Q(t)| = 5, 5^2$ or 5^3. If either of the latter two cases holds, then, as $|C_{S/Q}(t)| = 5$, $|C_S(t)| \geq 5^3$ and we are done. If $|C_Q(t)| = 5$, then we replace t by $t' = tz$. Then t' is an involution and $|C_Q(t')| = 5^3$ and we are done once again.

We now show that there is an involution $t \in N_H(S)$ with $|[S/Q, t]| = 5$. For this we use the fact that Q is contained in a torus K of order 5^8 in $E_8(4)$. Then (for some choice of K) $KS \in \mathrm{Syl}_5(G)$ and consequently $Z_2(S) = Z_2(KS)$. By Lemma 4, $C_G(Z_2(S)) = K$ and so $C_G(Q) = K$ also. Therefore, $K \trianglelefteq N_G(Q)$ and so $ST \leq N_G(K)$. Since $N_G(K)/K \cong W(E_8)$ we may calculate in $W = W(E_8)$. Let $R \in \mathrm{Syl}_5(W)$. Then $N_W(R)$ can be seen in the subgroup $(\mathrm{Sym}(5) \times \mathrm{Sym}(5)).2^2$ coming from the diagram $A_4 \times A_4$ (see [4]). Thus a Sylow 2-subgroup of $N_W(R)$ has order 2^6. Now we can identify $N_H(S)K/K$ with a subgroup of $N_W(R)$. Since TK/K has order 2^5, we see that $TK/K \cap N_W(R)$ contains the involutions from the subgroup $\mathrm{Alt}(5) \times \mathrm{Alt}(5)$ which we assume contains R. Finally, in this subgroup we see $\mathrm{Dih}(10) \times 5$ and we can choose t in this subgroup satisfying $[S,t] < S$ and $[S,t] \not\leq Q$. This concludes the proof of the lemma. ∎

Combining Proposition 5 and Lemma 6 with discussion before Lemma 6, we have the following result which is a substantial part of Theorem 2.

COROLLARY 7: *There is a unique conjugacy class of subgroups* $\mathrm{PSL}_4(5)$ *in* $E_8(4)$. *The normalizer in* $E_8(4)$ *of this group is isomorphic to the extension* $\mathrm{PSL}_4(5).2$ *of the simple group by a graph automorphism which centralizes a subgroup isomorphic to* $\mathrm{PSp}_4(5)$. ∎

Definition 2: Suppose that $R \cong \mathrm{PSp}_4(5)$ and let R_1 and R_2 be the two maximal parabolic subgroups containing a fixed Borel subgroup B of R. Then a pair of subgroups (P_1, P_2) of a group G is an amalgam of type $\mathrm{PSp}_4(5)$ provided, for $i = 1,2$, P_i has the same chief factors as R_i and, for $T \in \mathrm{Syl}_5(P_1 \cap P_2)$, $P_1 \cap P_2 = N_{P_1}(T) = N_{P_2}(T)$.

We may assume that $R_1 \cong 5_+^{1+2}.(\mathrm{SL}_2(5) \circ 4)$ and that $R_2 \cong 5^3 : \Omega_3(5).4 \cong 5^3 : \mathrm{Alt}(5).4$. We also note that $B/O_5(B) \cong 4 \times 2$.

PROPOSITION 8: *There is at most one conjugacy class of amalgams of type* $\mathrm{PSp}_4(5)$ *in* $E_8(2)$.

Proof: Let $G = E_8(2)$, $S \in \mathrm{Syl}_5(G)$ and use the notation from Proposition 3. Suppose that (P_1, P_2) is an amalgam of type $\mathrm{PSp}_4(5)$ contained in G. Let $T \in \mathrm{Syl}_5(P_1 \cap P_2)$ and assume without loss of generality that $T \leq S$. Then $T_3 = O_5(P_2) \leq T$ is elementary abelian of order 5^3. Notice that T_3 is a characteristic subgroup of T. Therefore, as $|S:T| = 5$, T_3 is normal in S and Proposition 3 (v) implies $T_3 = Q_3$. Thus Proposition 3 (vii) implies $P_2 \leq N_Y(Q_3)$. Notice that $N_Y(Q_3)/Q_3$ is a subgroup $(5 \times \mathrm{Alt}(5)).(2 \times 4)$. Therefore, $P_2'' = N_Y(Q_3)'' \cong 5^3.\mathrm{Alt}(5)$ is uniquely determined up to conjugacy in G. Let $T_1 \geq Q_4$ be such that $T_1 \trianglelefteq N_Y(Q_3)$ and $|T_1/Q_4| = 2$. Then, as T_1 inverts Q_4, T_1Q_3 is determined uniquely up to conjugacy in Q_4T_1. Define $P_2^* = N_Y(Q_3)''T_1$. Then $P_2^*/Q_3 \cong 2 \times \mathrm{Alt}(5)$. We have

(8.1) P_2^* is uniquely determined up to conjugacy in G. In particular, we may suppose that $P_2^* \leq P_2$.

Let K be a complement to T in $N_{P_2^*}(T)$. So $K \cong 2 \times 2$ and on $T/\Phi(T)$, K induces the matrix group $\langle \mathrm{diag}(-1,1), \mathrm{diag}(1,-1)\rangle$. Consequently K normalizes exactly two subgroups of T of order 5^3 which contain $\Phi(T)$. Let

$E_1 = O_5(P_1)$. Since K normalizes both Q_3 and E_1 and $E_1 \geq \Phi(T)$, we see that E_1 is uniquely determined by P_2. Furthermore, E_1 is extraspecial and so, setting $P_1^* = N_G(E_1)'$, $P_1^* \cong 5^{1+2} : \mathrm{SL}_2(5)$ by Proposition 3 (x). It follows that $P_1^* \leq P_1$. Let $K_1 = N_{P_1^*}(T)$. Then K_1/T is cyclic of order 4 and, as P_2^* determines P_1^*, it also determines K_1. Now $K_1 \leq P_1 \cap P_2$ and so we infer that $P_2 = K_1 P_2^*$ is uniquely determined. Finally, setting $K_2 = N_{P_2}(T)$, we have $P_1 = P_1^* K_2$ and the amalgam (P_1, P_2) is determined uniquely up to conjugacy in G. ∎

Definition 3: Suppose that $R \cong \mathrm{PSL}_3(5)$ and let R_1 and R_2 be the two maximal parabolic subgroups containing a fixed Borel subgroup B of R. Then a pair of subgroups (P_1, P_2) of a group G is an amalgam of type $\mathrm{PSL}_3(5)$ provided, for $i = 1, 2$, P_i has the same chief factors as R_i and, for $S \in \mathrm{Syl}_5(P_1 \cap P_2)$, $P_1 \cap P_2 = N_{P_1}(S) = N_{P_2}(S)$.

We remind the reader that $R_1 \cong R_2 \cong 5^2 : \mathrm{GL}_2(5)$.

PROPOSITION 9: *There is at most one conjugacy class of amalgams of type* $\mathrm{PSL}_3(5)$ *in* $\mathrm{E}_8(2)$.

Proof: Suppose that (P_1, P_2) is an amalgam of type $\mathrm{PSL}_3(5)$ in $G = \mathrm{E}_8(2)$. Use the notation from Proposition 3. Let $E \in \mathrm{Syl}_5(P_1 \cap P_2)$. Then E is extraspecial of order 125. Of course we may suppose that $E \leq S$ and so $Q_2 \leq E$ by Proposition 3 (ix). By Proposition 3 (xi), P_1 is determined uniquely up to conjugacy in G and of course $N_{P_1}(E)$ is determined up to conjugacy in P_1. Now $N_{P_1}(E) \cong 5^{1+2} : (4 \times 4)$ with Sylow 2-subgroup $T \cong 4 \times 4$ acting on $E/\Phi(E)$ as a diagonal group of automorphisms. Thus T normalizes precisely two elementary abelian subgroups of E of order 25. These groups must be $O_5(P_1)$ and $O_5(P_2)$. Let $t \in N_{N_G(E)}(TE)$ with $O_5(P_1)^t = O_5(P_2)$ and set $P_2^* = P_1^t$. Then (P_1, P_2^*) is an amalgam of type $\mathrm{PSL}_3(5)$. To complete the lemma we only have to show that $P_2^* = P_2$. This however follows from Proposition 3 (xi) as $P_2 \cap P_2^* \geq N_{P_1}(E)$. ∎

PROPOSITION 10: *There are subgroups of* $\mathrm{E}_8(2)$ *isomorphic to* $\mathrm{Aut}(\mathrm{PSL}_3(5))$ *and* $\mathrm{Aut}(\mathrm{PSp}_4(5))$.

Proof: Let $G = \mathrm{E}_8(4)$ and consider the subgroup $K \cong \mathrm{PSL}_4(5)$. Then $H = N_G(K)$ is isomorphic to $\mathrm{PSL}_4(5).2$ where the outer automorphism is the graph automorphism σ which centralizes a subgroup of K isomorphic to $\mathrm{PSp}_4(5)$. Then, by Proposition 5, H is determined uniquely up to conjugacy in G. Let τ be the Frobenius automorphism of G with fixed subgroup $E_8(2)$ and consider $G_1 = G\langle\tau\rangle$. Then, by the Frattini Argument, $G_1 = N_{G_1}(H)G$. So $[N_{G_1}(H) : H] = 2$. Set $H_1 = N_{G_1}(H)$. Since σ is not a diagonal automorphism of $\mathrm{PSL}_4(5)$, we infer that H_1/K is elementary abelian of order four. Furthermore, from the structure of $\mathrm{Out}(\mathrm{PSL}_4(5)) \cong \mathrm{Dih}(8)$ we infer that H_1/K has two elements corresponding to graph automorphisms (which are conjugate in $\mathrm{Out}(\mathrm{PSL}_4(5))$) and a diagonal automorphism. Let t_1K and t_2K be the two cosets of K in H_1 which are not contained in G. And assume that t_1 is the coset of the graph automorphism. Each of the cosets t_1H' and t_2H' intersects exactly two conjugacy classes of involutions of H_1 (see [5, Table 4.5.1]). Let the representatives of these classes be t_1 and t_1', t_2 and t_2' respectively. Now using [5, Table 4.5.1], up to change of notation, we have $C_H(t_1) \cong \mathrm{Aut}(\mathrm{PSp}_4(5))$, $O^{5'}(C_H(t_1')) \cong \mathrm{PSL}_2(5) \times \mathrm{PSL}_2(5)$, $C_H(t_2) \cong \mathrm{Aut}(\mathrm{PSL}_3(5))$ and $O^{5'}(C_H(t_2')) \cong \mathrm{PSL}_2(25)$. On the other hand, by Lang's Theorem t_1, t_1', t_2 and t_2' are all G-conjugate to τ (see [1, 19.1]). It follows that all the aforementioned centralizers are contained in subgroups of G isomorphic to $E_8(2)$. In particular, the statement in the proposition holds. ∎

There is a second existence proof of the subgroup $\mathrm{PSL}_3(5)$ in $E_8(2)$ which has the advantage that it also shows that $\mathrm{PSL}_3(5)$ is a subgroup of $E_8(p)$ for all primes $p \neq 5$ (for $p = 5$ the existence is clear). We explain the proof in characteristic 2 and continue the notation of Proposition 10. So we have $G_1 = G\langle\tau\rangle$. By [3] there is a unique conjugacy class of subgroups $P = 5^3 : \mathrm{PSL}_3(5)$ in G and so $[N_{G_1}(P) : P] = 2$. Since $\mathrm{GL}_3(5) \cong 4 \times \mathrm{PSL}_3(5)$ and $Z(N_G(P)) = 1$, we infer that there is an involution t in $N_{G_1}(P)$ with $C_P(t) \cong \mathrm{PSL}_3(5)$. But then Lang's Theorem implies that $\mathrm{PSL}_3(5)$ is isomorphic to a subgroup of $E_8(2)$.

Proof of Theorem 1: The existence of subgroups of $E_8(2)$ which are isomorphic to $\mathrm{PGSp}_4(5)$ and $\mathrm{Aut}(\mathrm{PSL}_3(5))$ and their uniqueness up to conjugacy is established by combining Propositions 8, 9 and 10. Finally we observe that these subgroups are maximal in $E_8(2)$. Write L for either of the two simple subgroups, and let M be an overgroup of L maximal in G. Let N be a minimal normal subgroup of M. Assume first that N is a product of non-abelian simple groups, each isomorphic to the simple group S. If S is of Lie type in characteristic 2, then it has Lie rank at most 8. Since L has no representation over the field of two elements of dimension less than 24 [8] (note that the 12-dimensional representation requires the field of order 4), this is impossible by [12]. Also, the only possibility for S not of Lie type in characteristic 2 is $\mathrm{PSL}_4(5)$ by [12], and that is only a subgroup of $E_8(2^a)$ if a is even. Thus $N = L$ follows in this case, proving the desired maximality. Hence we can now assume that N is elementary abelian and L is involved in M/N. This is clearly impossible: as above, L is not in a centralizer in $E_8(2)$, since these involve only subsystem subgroups. Hence N would have to be a 5-subgroup, and this is clearly impossible. ∎

Proof of Theorem 2: Let $G = E_8(4)$. The existence of a subgroup K of G isomorphic to $PSL_4(5)$ and the structure of $N = N_G(K)$ is given in Corollary 7. So to complete the proof of Theorem 2 we have to show that N is a maximal subgroup of G. Assume that $L < G$ contains K. By [12] K is not contained in a parabolic subgroup or a subsystem subgroup of G. In particular, $C_G(K) = 1$. Let M be a minimal normal subgroup of L. Assume that M is elementary abelian. Since M is not contained in a parabolic of G, we see that M is not a 2-group. Now the minimal p-modular representation of K has dimension 5 for $p = 5$ and, by [6, 7], 154 if p is not 5. It follows that M is not a p-group. Therefore M is a direct product of non-abelian simple groups. Since the minimal faithful permutation degree of K is 156, we infer that $L \leq M$ and M is simple. We now deduce that $M = L$ by applying [12]. Thus $L = N$ and we are done. ∎

Remarks:

(i) There are at least two conjugacy classes of $PSp_4(5)$ in $E_8(4)$, one obtained via $PSL_4(5)$ and the other via the parabolic subgroup D_7 (cf. [12]). It is however possible that the class of primitive subgroups (i.e., not contained in a parabolic subgroup) in $E_8(4)$ is unique.

(ii) It is shown in [12, 8.5] that $PSp_4(5)$ cannot be a subgroup of $E_8(q)$ if q is not a power of 2 or 5. On the other hand, our alternative proof of the embedding of $PSL_3(5)$ into $E_8(2)$ yields an embedding of $PSL_3(5)$ into $E_8(p)$ for any p other than 5. The following character calculation shows that $\mathrm{Aut}(PSL_3(5))$ is not a subgroup of $E_8(p)$ whenever $p \neq 2$ or 5. Let V be the adjoint module for $E_8(p)$ with $p \neq 2$ or 5. From [12, Table 4] the character of any element of order 2 on V is either 24 or -8. Thus the character tables in [4] and [8] show that the restriction of V to a subgroup $PSL_3(5)$ has exactly two composition factors each of dimension 124. Furthermore, if $H = \mathrm{Aut}(PSL_3(5)) \leq E_8(p)$, then the outer automorphism of H exchanges the two composition factors for H' and so has character 0 on V. Hence it is impossible for H to be a subgroup of $E_8(p)$.

(iii) Matrices generating $E_8(2)$ in its adjoint representation are available from the ATLAS website [14]. Using this representation we have constructed $PSL_3(5)$ and $PSp_4(5)$ as subgroups of $E_8(2)$. See http://web.mat.bham.ac.uk/C.W.Parker/L35 (PSp45) for matrices generating these subgroups. We have decomposed the adjoint module for $E_8(2)$ for each of $PSL_3(5)$ and $PSp_4(5)$. The first restricts as a uniserial module $\substack{124 \\ 124}$ and the second restricts as $40 \oplus \substack{104 \\ 104}$.

(iv) There are exactly two representations for $PSL_4(5)$ of dimension 248 defined over a field of characteristic 2 [6, 7]. These representations are both contained as composition factors in the GF(2) permutation module on the stabiliser of a line in the natural 4-dimensional representation of $SL_4(5)$. By restricting to a subgroup $PSL_3(5)$, we can identify which of these two representations is isomorphic to the restriction of the adjoint $E_8(4)$-module and then we may calculate the restriction to all the subgroups of $E_8(2)$ discovered in Proposition 10. We list these restrictions in Table 1.

Table 1

Centralizer	Decomposition
$C_G(t_1)$	40/104/104
$C_G(t_1')$	32/8/1/8/8/32/8/1/8/16/1/8/1/8/16/1/1/1/8/1/32/8/8/32
$C_G(t_2)$	124/124
$C_G(t_2')$	24/2/1/24/2/24/12/1/12/2/24/26/1/12/2/24/26/2/2/1/24

ACKNOWLEDGEMENTS: This paper was written while the authors were visiting CalTech in the autumn of 2003. It is a pleasure to thank the department and in particular Michael Aschbacher for their hospitality. We also thank J.-P. Serre for helpful comments on an earlier version of this paper.

References

[1] M. Aschbacher and G. M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Mathematical Journal **63** (1976), 1–91.

[2] R. W. Carter, *Finite Groups of Lie Type. Conjugacy Classes and Complex Characters*, Reprint of the 1985 original, Wiley Classics Library, A Wiley-Interscience Publication, John Wiley & Sons, Ltd., Chichester, 1993.

[3] A. M. Cohen, M. W. Liebeck, J. Saxl and G. M. Seitz, *The local maximal subgroups of exceptional groups of Lie type, finite and algebraic*, Proceedings of the London Mathematical Society (3) **64** (1992), 21–48.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups,* With computational assistance from J. G. Thackray, Oxford University Press, Eynsham, 1985.

[5] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups,* Number 3, Part I, Chapter A, *Almost simple K-groups,* Mathematical Surveys and Monographs, 40.3, American Mathematical Society, Providence, RI, 1998.

[6] G. Hiss and G. Malle, *Low-dimensional representations of quasi-simple groups,* London Mathematical Society Journal of Computational Mathematics **4** (2001), 22–63.

[7] G. Hiss and G. Malle, *Corrigenda: "Low-dimensional representations of quasi-simple groups",* London Mathematical Society Journal of Computational Mathematics **5** (2002), 95–126.

[8] C. Jansen, K. Lux, R. Parker and R. Wilson, *An Atlas of Brauer Characters,* Appendix 2 by T. Breuer and S. Norton, London Mathematical Society Monographs, New Series, 11, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[9] P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups,* London Mathematical Society Lecture Note Series, 129, Cambridge University Press, Cambridge, 1990.

[10] M. W. Liebeck and J. Saxl, *On the orders of maximal subgroups of the finite exceptional groups of Lie type,* Proceedings of the London Mathematical Society (3) **55** (1987), 299–330.

[11] M. W. Liebeck, J. Saxl and G. M. Seitz, *Subgroups of maximal rank in finite exceptional groups of Lie type,* Proceedings of the London Mathematical Society (3) **65** (1992), 297–325.

[12] M. W. Liebeck and G. M. Seitz, *On finite subgroups of exceptional algebraic groups,* Journal für die reine und angewandte Mathematik **515** (1999), 25–72.

[13] C. Parker and J. Saxl, *Localization and finite simple groups,* Israel Journal of Mathematics, this volume.

[14] R. A. Wilson et al., ATLAS *of Group Representations,* http://web.mat.bham.ac.uk/atlas/v2.0/ (2004).